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On isotropic tensors

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1. An isotropic tensor is one the values of whose components are unaltered by any rotation of rectangular axes (with metric $\Sigma_i(dx_i)^2$). Those up to order 4 in 2 and 3 dimensions have many applications. The results suggest a general theorem for tensors of order m in n dimensions, that any isotropic tensor can be expressed as a linear combination of products of δ and ϵ tensors, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, and $\epsilon_{i_1 \ldots i_n}$ is 0 if any two of the i_1 to i_n are equal, 1 if $i_1 \ldots i_n$ is an even permutation of $1, 2, 3, \ldots, n$, and -1 if it is an odd permutation.

For $n > 3$ a rotation is not necessarily about any single axis. We define it as an orthogonal transformation of determinant $+1$. This can be shown to be expressible as the result of at most $\frac{1}{2}n(n-1)$ successive rotations on the coordinate planes, and therefore it is enough to consider such rotations separately. This is stated without proof, among other cases, by D. E. Littlewood ((4), p. 18). I have not seen a proof in print. Dr G. A. Reid has given me one. Consider a rotation through α on the 12 plane. The matrix l_{ij} becomes

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 & \ldots \\ -\sin\alpha & \cos\alpha & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \end{pmatrix} \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ \ldots & \ldots & \ldots \end{pmatrix}$$
$$= \begin{pmatrix} \cos\alpha l_{11} + \sin\alpha l_{21} & \cos\alpha l_{12} + \sin\alpha l_{22} & . \\ -\sin\alpha l_{11} + \cos\alpha l_{21} & -\sin\alpha l_{12} + \cos\alpha l_{22} & . \\ \ldots & \ldots & l_{33} \end{pmatrix}.$$

Take $\tan\alpha = -l_{12}/l_{22}$; then l_{12} is annihilated. It may be convenient to take α so that $l'_{11} \geqslant 0$. Then we proceed to remove $l_{13} \ldots l_{1n}$. Since the sum of squares of elements of a row or column in an orthogonal matrix is 1, $l'_{11} = 1$, and all of $l'_{21} \ldots l'_{n1} = 0$. Thus by $n-1$ rotations on the coordinate planes $\mathbf{l}$ is reduced to a form with $l'_{11} = 1$; the other elements with either $i = 1$ or $j = 1$ are 0, and the rest form an orthogonal matrix of order $n-1$. By repetition $\mathbf{l}$ is reduced by $\frac{1}{2}n(n-1)$ rotations to one with all diagonal l_{ij} equal to δ_{ij}; multiplying by the reciprocal of this set expresses l_{ij} as a product of δ_{ij} and a set of rotations on the coordinate planes.

It might apparently happen that l'_{rr} (not summed) at some stage cannot be made positive, and must be -1. But if, say, l'_{22} and l'_{33} are both -1 a rotation of π on the 23 plane would make them $+1$. If only one diagonal element remains -1 the matrix represents a reflexion, which we are not considering.

2. *The main theorem is as follows. If a tensor* $u_{i_1 \ldots i_m}$ *is transformed by a rotation* $\mathbf{l}$, *giving*

$$u'_{j_1 \ldots j_m} = l_{i_1 j_1} \ldots l_{i_m j_m} u_{i_1 \ldots i_m},$$

then $u'_{j_1\ldots j_m} = u_{j_1\ldots j_m}$, *if and only if it is expressible as a linear combination of products of the form* $\delta_{i_r i_s}$ *and possibly one* ϵ *factor.*

The condition is obviously sufficient. Necessity can be proved as follows.

We call the i_r S suffixes. Each can take the values 1 to n, and when numerical values are given, we call them N suffixes.

Rotate through π on the 12 plane. The matrix $\mathbf{l}$ for the transformation is

$$l_{11} = l_{22} = -1, \quad l_{12} = l_{21} = 0,$$

other l_{ij} being δ_{ij}. If in a component N suffix 1 occurs k times and 2 occurs l times, the component is multiplied by $(-1)^{k+l}$. Hence if no component is altered it must be 0 or $k+l$ must be even. This can be applied to any pair of N suffixes; thus non-zero components separate into two classes, in one of which every N suffix occurs an even number of times, and in the other an odd number of times.

Rotate through $\frac{1}{2}\pi$ on the 12 plane. The matrix for the transformation is

$$l_{11} = l_{22} = 0, \quad l_{12} = -1, \quad l_{21} = 1;$$

other diagonal components δ_{ij}. Write the component as $u(k, l))\,v$, where v is unaltered by this transformation. Then the transformed component is $(-1)^k u(l, k)\,v$. This will be equal to $u(k, l)\,v$ provided either k is even and $u(l, k) = u(k, l)$, or if k is odd and $u(l, k) = -u(k, l)$. In the first case transposition of two suffixes leaves the component unaltered; in the second it reverses it. By the last result, or by rotation through $-\frac{1}{2}\pi$, if k is even, so is l, and if k is odd, so is l.

If k and l are odd every N suffix must occur an odd number of times in every non-zero component. If for instance N suffixes 1 and 2 occurred an odd number of times, and 3 an even number, a rotation through $\frac{1}{2}\pi$ on the 13 plane would reverse the sign.

3. Any N suffix that occurs in a component occurs in places corresponding to a particular set of S suffixes. Under any rotation through $\frac{1}{2}\pi$ on a coordinate plane, two N suffixes are interchanged, with or without reversal of sign, but there is no change of the S suffixes that they replace. Thus the distribution of the sets of equal N suffixes specifies a partition of the S suffixes, such that all of a set take the same value of an N suffix. Different partitions transform independently (e.g. no rotation can turn 1122 into 1313). Thus every such partition specifies an independent set of equal components and is specified by which of the i_r take equal N values. Consider then partitions so specified. The original tensor is a linear combination of them.

In the case of even k (when m must be even) consider components with $i_1 = i_2 = 1$ or $i_1 = i_2 = 2$. The remaining i_r can taken any values up to n. Rotation on the 12 plane does not alter the components, and so long as none of i_3 to i_m is 1 or 2 they are unaltered by any rotation on the rs planes where $i_r, i_s > 1, 2$. Thus all these components are of the form $\delta_{i_1 i_2} v_{i_3\ldots i_m}$, with $i_1, i_2 = 1, 2$, where $\mathbf{v}$ is an isotropic tensor of order $m-2$ in $n-2$ dimensions. In some components of the original tensor some of the i_r for $r > 2$ may also be 1 or 2; but consider rotations on planes $1r$, $2r$ with $r > 2$. If $\mathbf{u}$ is still isotropic the components are unaltered. If not, such rotations still define a possible form of $\mathbf{u}$, say $\mathbf{u}_0$, which we may subtract from the original $\mathbf{u}$, leaving $\mathbf{w}$. This will leave components zero unless, for instance, $i_1 i_2 i_3 i_4$ are all 1 or all 2. Suppose there are A components; then rotating through $\frac{1}{4}\pi$ would give (apart from an unchanged factor)

$$w'_{1111} = 2\left(\frac{1}{\sqrt{2}}\right)^4 A = \tfrac{1}{2}A, \quad w'_{1122} = w'_{2211} = \tfrac{1}{2}A, \quad w'_{2222} = \tfrac{1}{2}A$$

instead of A, 0, 0, A. Thus **w** is not isotropic for rotations on the 12 plane. Hence **u** is not isotropic in n dimensions. Then, by repetition, **u** is the product of δ tensors. A similar argument will apply if an N suffix occurs a larger even number of times. We could have begun with components where equal N suffixes are not cases of i_1, i_2 and hence any isotropic tensor with even k is a linear combination of products $\delta_{i_r i_s}$.

Since reflexion does not alter components with even k this result is also correct for orthogonal transformations that include reflexions.

Alternatively, consider the case $n = 2$ and the zero component $u_{111\ldots 2}$, where 1 occurs $2k-1$ times and 2 once. Make a small rotation $l_{ij} = \delta_{ij} + \eta_{ij}$, where $\eta_{11} = \eta_{22} = 0$, $\eta_{12} = \alpha$, $\eta_{21} = -\alpha$, other $\eta_{ij} = 0$. Then

$$u'_{111\ldots 2} = \Pi(\delta_{j_r j_r} + \eta_{i_r j_r})\, u_{i_1 i_2 \ldots i_{2k}}$$

with $j_1 \ldots j_{2k-1} = 1$, $j_{2k} = 2$. If all pairs i_r, j_r are equal the contribution is the original component, which is zero. The terms linear in α give

$$\alpha(u_{2111\ldots 2} + u_{12\ldots 2} + \ldots + u_{111\ldots 22}) - \alpha u_{111\ldots 1} = 0.$$

Hence $u_{111\ldots 1} = u_{2111\ldots 2} + u_{12\ldots 2} + \ldots + u_{111\ldots 22}$. Since all terms in the brackets are components of independent tensors, we can put all but the first zero, and it will follow that if this is a product of δ's so is $u_{111\ldots 1}$. The same applies to other components.

If $n > 2$ the result follows by a small rotation on the 12 plane keeping the other axes fixed.

This result for $n = 3$ and $m = 4$ is derived in this way in *Cartesian Tensors* ((1), eq. 14, p. 69). It is found in *Methods of Mathematical Physics* (2) by constructing a scalar polynomial which must, for $m = 4$, be $(x_1^2 + x_2^2)^2$, and the result is found by differentiation.

4. In the case of odd k, *every* N suffix must occur an odd number of times in every non-zero component (since 0 is an even number). In each set of S suffixes that take the same N value, pick out the first, which we shall call the E suffix. Under rotation each remains the first in the set of equal N suffixes. Apart from the E suffixes every N suffix occurs an even number of times in every component, and its contribution in a $\frac{1}{2}\pi$ rotation does not alter the component, and the signs will be all the same or all opposite to those of the components of $\epsilon_{i_1 \ldots i_n}$. The components are therefore $\boldsymbol{\epsilon}$ times a set of quantities, say $v_{i_{n+1} \ldots i_m}$. Contrast with $\epsilon_{i_1 \ldots i_n}$. This must give an isotropic tensor; but we see that this is $n!\, v_{i_{n+1} \ldots i_n}$ which must therefore be an isotropic tensor with every N suffix occurring an even number of times, and therefore a linear combination of δ products. This completes the theorem.

It follows that for non-zero isotropic tensors every $m < n$ must be even, and conversely an ϵ factor can arise only for $m \geqslant n$.

5. No isotropic tensor need contain more than one ϵ factor. For the product

$$\epsilon_{i \ldots i_n}\, \epsilon_{k \ldots k_n} = (0,\ 1 \text{ or } -1)$$

according as any pair of i's or k's are equal, or if all i's and all k's are different and both derived from $1, 2, \ldots, n$ by an even or both by an odd permutation, or if one is derived by an even and the other by an odd permutation. Now

$$\delta_{i_p k_q} - \delta_{i_q k_p} = \begin{cases} 1 & \text{if} \quad i_p = k_q \neq i_q, k_p, \\ -1 & \text{if} \quad i_q = k_p \neq i_p, k_q, \\ 0 & \text{otherwise.} \end{cases}$$

Multiply by the remaining δ_{ik} and we shall have $+1$ if the i and k are both odd or both even permutations of $1, 2, \ldots, n$, -1 if one is odd and the other even. Hence the product of two ϵ's can be expressed in terms of δ's. In 3 dimensions we have

$$\epsilon_{ikm}\epsilon_{jln} = \delta_{ij}\delta_{kl}\delta_{mn} - \delta_{ij}\delta_{kn}\delta_{ml} + \delta_{il}\delta_{kn}\delta_{mj} - \delta_{il}\delta_{kj}\delta_{mn} + \delta_{in}\delta_{kj}\delta_{ml} - \delta_{in}\delta_{kl}\delta_{mj},$$

which Vicente and I (3) have found useful in calculating elastic energies in a stressed sphere when the displacement is everywhere at right angles to the radius vector.

6. Weyl (11) has given a result equivalent to the main theorem but by a rather indirect method and not stated in this form.

The complete theorem was given by M. Pastori (5–10) and the methods are largely, but not entirely, similar to mine. I am grateful to Prof. F. Mainardi for the references.

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REFERENCES

(1) JEFFREYS, H. *Cartesian tensors* (Cambridge University Press, 1931).

(2) JEFFREYS, H. and JEFFREYS, B. S. *Methods of mathematical physics* (Cambridge University Press, 1946).

(3) JEFFREYS, H. and VICENTE, R. O. *Acad. Roy. Belg., Bull. Cl. Sci.* (5), **53** (1967), 926–33.

(4) LITTLEWOOD, D. E. *The theory of group characters* (1940).

(5) PASTORI, MARIA. *Atti. Acc. Naz. Lincei Rendiconti, Cl. Sci. Fis. Mat. Natur. VI*, **12** (1930), 374–379.

(6) PASTORI, MARIA. *Atti. Acc. Naz. Lincei Rendiconti, Cl. Sci. Fis. Mat. Natur. VI*, **12** (1930), 499–502.

(7) PASTORI, MARIA. *Atti. Acc. Naz. Lincei Rendiconti, Cl. Sci. Fis. Mat. Natur. VI*, **13** (1931), 119–114.

(8) PASTORI, MARIA. *Atti. Acc. Naz. Lincei Rendiconti, Cl. Sci. Fis. Mat. Natur. VI*, **17** (1933), 439–443.

(9) PASTORI, MARIA. *Scritti matematici offerti a Luigi Berzolari*, pp. 223–238. (Pavia, 1936.)

(10) PASTORI, MARIA. *Boll. Un. Mat. Ital.* 2 (1939).

(11) WEYL, H. *The classical groups.* (1939), pp. 52–66, 137–163.